Heat Eq. (part 3)

$u_t = k u_{xx} \qquad 0 < x < L \qquad t > 0$

now we will insulate the ends (like w/ the lateral surface)

heat cannot leave the rod through the ends anymore.

$\rightarrow \quad \dfrac{\partial u}{\partial x} = 0$ at $x=0$ and $x=L$

$$\left.\begin{array}{l} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{array}\right\} \text{Neumann boundary conditions}$$

same initial condition as before

$u(x,0) = f(x)$

solve by using separation of variables

$$u(x,t) = X(x)\,T(t)$$

$$u_t = X T' \qquad u_{xx} = X'' T$$

$$u_t = k u_{xx}$$

$$X T' = k X'' T$$

separate: $$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$ same separation constant

two ODE's: $$X'' + \lambda X = 0$$
$$T' + k\lambda T = 0$$

BC's: $u_x(0,t) = 0 \to X'(0)T(t) = 0 \to X'(0) = 0$

$u_x(L,t) = 0 \to X'(L)T(t) = 0 \to X'(L) = 0$

solve for X:

usual solution: $X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$ $\quad (\lambda \neq 0)$

$$X'(x) = -\sqrt{\lambda} A \sin(\sqrt{\lambda} x) + \sqrt{\lambda} B \cos(\sqrt{\lambda} x)$$

$$X'(0) = 0 = \sqrt{\lambda} B \rightarrow B = 0$$

$$X'(L) = 0 = -\sqrt{\lambda} A \sin(\sqrt{\lambda} L)$$ require $A \neq 0$

so, $\sin(\sqrt{\lambda} L) = 0$

so, $\sqrt{\lambda} L = n\pi$ $\quad n = 1, 2, 3, \ldots$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$ eigenvalues (same as before)

solution ~~for~~ for each n is

$$X_n = \cos(\sqrt{\lambda} x)$$ (drop A scaling constant)

$$\boxed{X_n = \cos\left(\frac{n\pi}{L} x\right)}$$ eigenfunctions cosines this time

back to $X'' + \lambda X = 0$ $\quad X'(0) = X'(L) = 0$

what if $\lambda = 0$?

$$X'' = 0 \rightarrow X(x) = Ax + B \qquad \text{BC's: } X'(0) = X'(L) = 0$$

$$X'(x) = A$$

$$X'(0) = X'(L) = 0 \longrightarrow A = 0$$

so, X = constant also meets the BC's

$\boxed{X = 1}$ for the case when $\lambda = 0$

(drop the B scaling constant)

notice $\lambda = 0 \rightarrow$ eigenvalue is 0

notice $\lambda = 0$ and $X = 1$ are included in $\lambda_n = \frac{n^2\pi^2}{L^2}$

and $X_n = \cos\left(\frac{n\pi}{L}x\right)$

so, $\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$ $\boxed{n = 0, 1, 2, 3, \ldots}$

$\boxed{X_n = \cos\left(\frac{n\pi}{L}x\right)}$

now we solve $T' + k\lambda T = 0$ (same as before)

$$\boxed{T_n = e^{-kn^2\pi^2 t/L^2}}$$

$n = 0, 1, 2, 3 \ldots$

for each n, solution is $u_n = X_n T_n = e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$

general solution is linear combination of all

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

let's separate out $n=0$ (A_0)

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

looks like a cosine series!

let's give $n=0$ an extra $\frac{1}{2}$ factor

IC: $(t=0)$

$u(x,0) = f(x)$

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

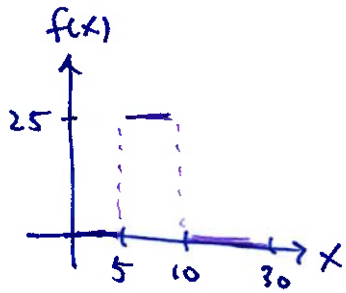
cosine series!

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

$n = 0, 1, 2, 3, \ldots$

example $L=30$, $K=1$, insulated ends

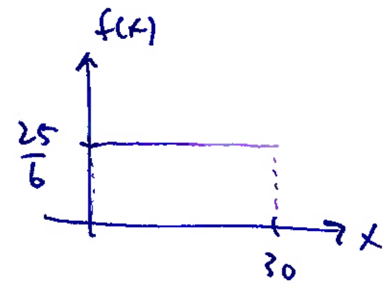
initial heating profile $f(x)=\begin{cases} 0 & 0<x<5 \\ 25 & 5<x<10 \\ 0 & 10<x<30 \end{cases}$



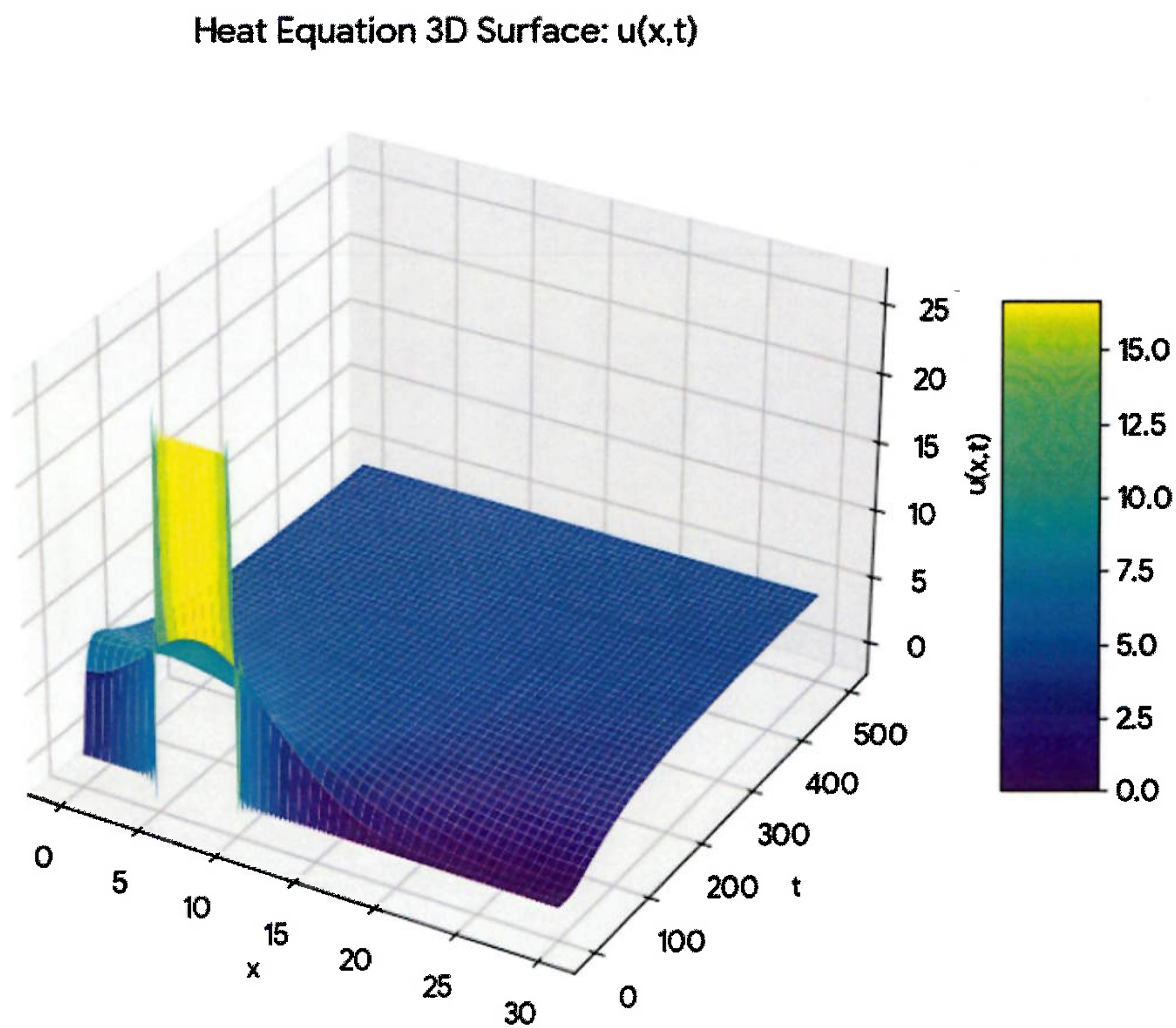
$$u(x,t)=\frac{25}{6}+\sum_{n=1}^{\infty}\frac{50}{n\pi}\left[\sin\left(\frac{n\pi}{3}\right)-\sin\left(\frac{n\pi}{6}\right)\right]e^{-n^2\pi^2 t/900}\cos\left(\frac{n\pi x}{30}\right)$$

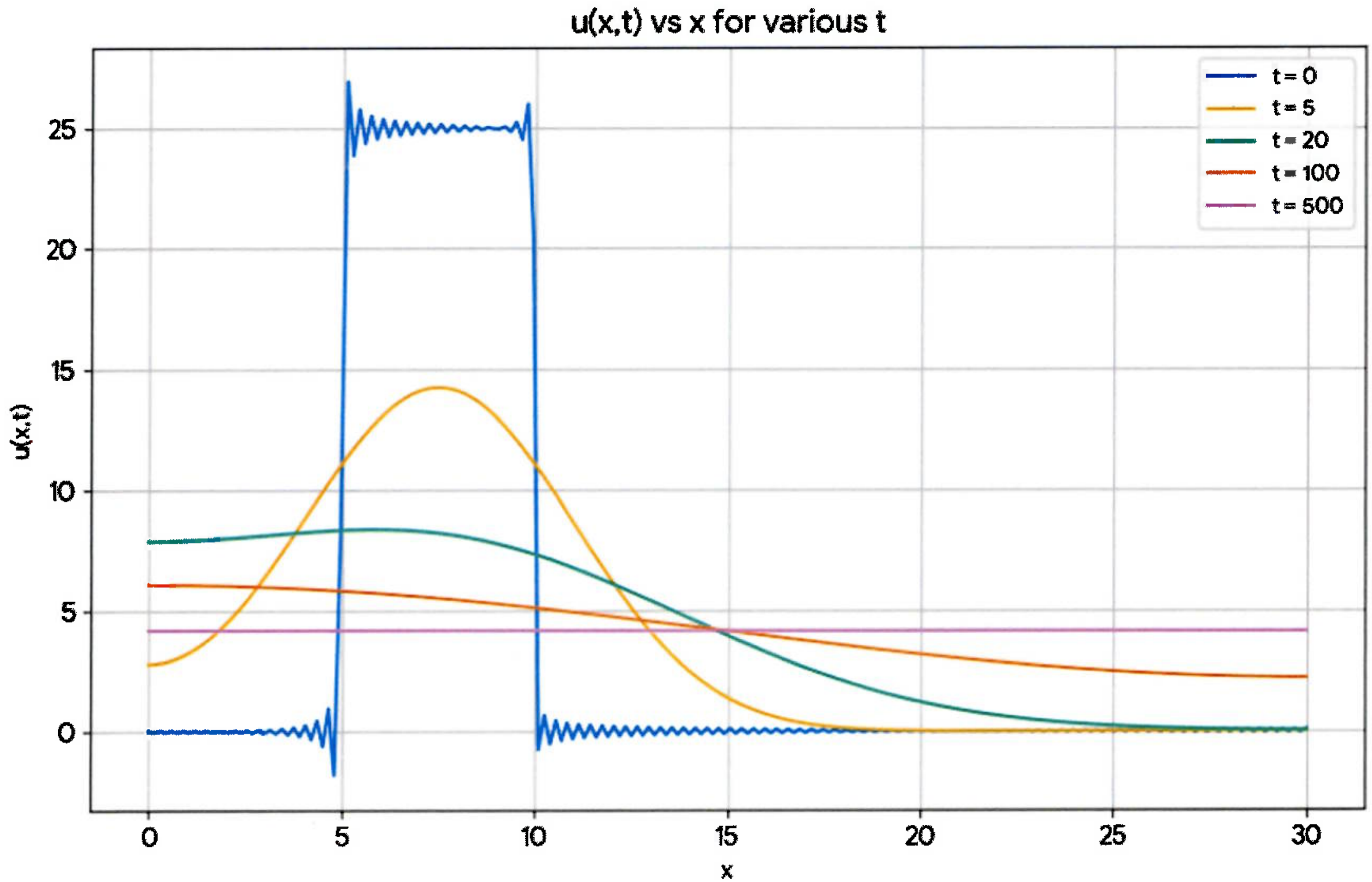
steady-state solution ($t\to\infty$)

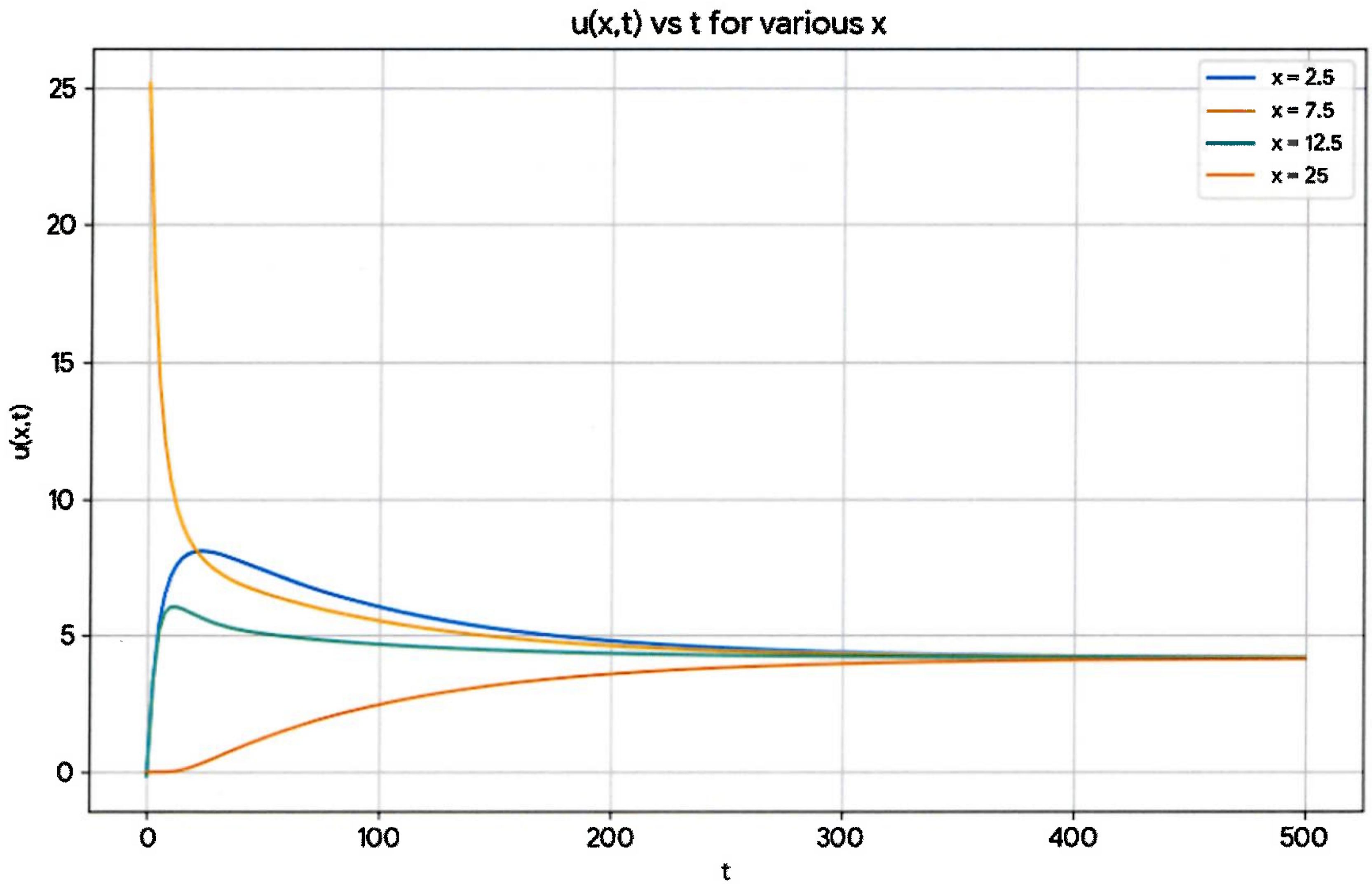
$u(x,t)=\frac{25}{6}$ (constant)



note: the area under the initial profile = area under the steady-state solution because heat cannot leave the rod, and heat doesn't want any spot to be hotter/colder than the avg. value near it and no ~~con~~ concavity


Heat Equation 3D Surface: u(x,t)
x
0
5
10
15
20
25
30
t
0
100
200
300
400
500
u(x,t)
0
5
10
15
20
25
0.0
2.5
5.0
7.5
10.0
12.5
15.0


u(x,t) vs x for various t
t = 0
t = 5
t = 20
t = 100
t = 500
u(x,t)
x
0
5
10
15
20
25
30


u(x,t) vs t for various x
x = 2.5
x = 7.5
x = 12.5
x = 25
25
20
15
10
5
0
u(x,t)
0
100
200
300
400
500
t